ON THE DIFFERENTIATION OF DEFINITE INTEGRALS

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THE object of this paper is to give a simpler proof of the theorem that

$$(A) \qquad \frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial f}{\partial \alpha}\,dx + f(b,\alpha)\,\frac{db}{d\alpha} - f(a,\alpha)\,\frac{da}{d\alpha}$$

than those that are current.

1. The Common Proofs. The theorem is usually proven by writing

$$\phi(\alpha) = \int_a^b f(x,\alpha)\,dx,$$

forming the difference:

$$\phi(\alpha+\Delta\alpha) - \phi(\alpha) = \int_{a+\Delta a}^{b+\Delta b} f(x,\alpha+\Delta\alpha)\,dx - \int_a^b f(x,\alpha)\,dx$$

$$= \left(\int_{a+\Delta a}^{a} + \int_a^b + \int_b^{b+\Delta b}\right) f(x,\alpha+\Delta\alpha)\,dx - \int_a^b f(x,\alpha)\,dx$$

$$= \int_a^b [f(x,\alpha+\Delta\alpha) - f(x,\alpha)]\,dx + \int_b^{b+\Delta b} f(x,\alpha+\Delta\alpha)\,dx$$

$$- \int_a^{a+\Delta a} f(x,\alpha+\Delta\alpha)\,dx,$$

and applying the theorems of mean value to these last three integrals; suitable assumptions being made about the continuity of the functions that enter. Cf., for example, Goursat-Hedrick, *Mathematical Analysis*, vol. I, §97.

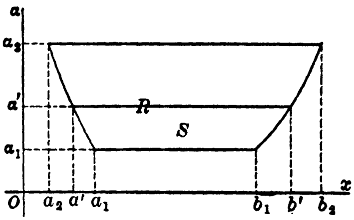
Another proof consists in changing the variable of integration so that the new limits of integration become constant:

$$t = \frac{x-a}{b-a}, \qquad x = (b-a)t + a,$$

$$\int_a^b f(x,\alpha)\,dx = (b-a)\int_0^1 f(x,\alpha)\,dt,$$

this latter case having been treated previously. Cf. Picard, *Traité d'analyse*, vol. I, chap. I, §18.

2. Critique of these Proofs. We will first state the theorem in detail in its simplest and most useful form. Let R be a region of the (x, α)-plane bounded by the right lines



$$\alpha = \alpha_1, \qquad \alpha = \alpha_2, \qquad (\alpha_1 < \alpha_2),$$

and the curves

$$a = \psi(\alpha), \qquad b = \omega(\alpha),$$

where each of the functions $\psi(\alpha)$ and $\omega(\alpha)$ shall be continuous, together with its first derivative, throughout the interval $\alpha_1 \leqq \alpha \leqq \alpha_2$, and where

$$\psi(\alpha) < \omega(\alpha).$$

The region R shall include its boundary.

The function $f(x, \alpha)$ shall be continuous at all points of R. Its partial derivative

$$\frac{\partial f}{\partial \alpha} = f_\alpha(x, \alpha)$$

shall exist and be continuous at all interior points of R, and this function, $f_\alpha(x, \alpha)$, shall remain finite throughout the interior of R.

Under the above conditions the integral

$$\int_a^b f(x, \alpha)\,dx$$

defines a function of α *having a derivative given by the above formula,* (A).

Turning now to the first of the proofs cited in §1 we see that if, in the neighborhood of a point $\alpha = \alpha'$ of the interval $\alpha_1 \leqq \alpha \leqq \alpha_2$, the functions $\psi(\alpha)$ and $\omega(\alpha)$ are both monotonic, the above transformation of the difference of the integrals, or a suitable modification of this transformation, is legitimate and the proof is sound. But there are several cases to consider when Δa and Δb change sign with $\Delta \alpha$.

If, in particular, the functions $f(x, \alpha)$, $f_\alpha(x, \alpha)$ are continuous throughout a larger region R' lying between the same parallels $\alpha = \alpha_1$ and $\alpha = \alpha_2$ and containing all the interior and boundary points of R situated between these

lines in its interior, the single transformation of the difference given above will hold for both positive and negative values of $\Delta\alpha$. But the theorem thus restricted is too narrow for all the ordinary applications of practice.

There are, then, even when $\psi(\alpha)$ and $\omega(\alpha)$ are both monotonic, several cases of the transformation of the difference of the above integrals to be considered.

If, however, one of the functions $\psi(\alpha)$, $\omega(\alpha)$ is not monotonic; for example, if

$$\omega(\alpha) = (\alpha - \alpha')^3 \sin \frac{1}{\alpha - \alpha'}, \qquad \alpha \neq \alpha',$$
$$\omega(\alpha') = 0,$$

there is trouble. The theorem is still true, but a special ϵ-proof is necessary.

The second proof is not open to this objection, but it turns out that it is necessary to assume the existence of the partial derivative of $f(x, \alpha)$ with respect to x, and thus the proof does not apply to the theorem stated in the above generality.*

3. A New Proof. We can obtain a simple proof of the theorem as follows. Consider the double integral

$$(1) \qquad I = \int_S\int f_\alpha(x, \alpha)dS,$$

extended over the region

$$S: \qquad a \leqq x \leqq b, \qquad \alpha_1 \leqq \alpha \leqq \alpha',$$

when $\alpha_1 < \alpha' \leqq \alpha_2$. We can first evaluate I by means of the iterated integral

$$(2) \qquad I = \int_{\alpha_1}^{\alpha'} d\alpha \int_a^b f_\alpha(x, \alpha)dx.$$

Secondly, we can evaluate I by means of Green's Theorem:

$$\int_S\int f_\alpha(x, \alpha)\,dS = -\int_C f(x, \alpha)dx.$$

* By means of this transformation, however, a simple proof can be given that

$$\int_a^b f(x, \alpha)dx$$

is a continuous function of α if $f(x, \alpha)$ is continuous in R, and a, b are continuous functions of α. The existence of a derivative with respect to x is here unnecessary.

Hence

$$I = -\int_{a_1}^{b_1} f(x, \alpha_1)\,dx - \int_{\alpha_1}^{\alpha'} f(b, \alpha)\,\frac{db}{d\alpha}\,d\alpha + \int_{a'}^{b'} f(x, \alpha')\,dx + \int_{\alpha_1}^{\alpha'} f(a, \alpha)\,\frac{da}{d\alpha}\,d\alpha. \tag{3}$$

Equating the two expressions (2) and (3) for I, dropping the accent against the α, and differentiating the equation thus resulting with respect to α, we obtain the theorem contained in the formula (A).

Instead of using the region S between the parallels $\alpha = \alpha_1$ and $\alpha = \alpha'$, we might have chosen an arbitrary value $\alpha_3 \neq \alpha'$. The two evaluations of I would have yielded the same final equation, the subscript 1 being merely replaced by 3. Thus, in particular, if $\alpha_3 = \alpha_2$, the excluded value $\alpha' = \alpha_1$ no longer presents an exception.

4. Generalizations. Corresponding to more general forms of Green's Theorem we obtain the theorem under consideration with less restrictive hypotheses. Thus if $f_\alpha(x, \alpha)$ is continuous within R, but does not remain finite on the boundary, the function $f(x, \alpha)$ still being assumed continuous in R, and if the surface integral (1) converges when extended over any region

$$S': \qquad a \leqq x \leqq b, \qquad \bar{\alpha}_1 \leqq \alpha \leqq \bar{\alpha}_2,$$

where $\alpha_1 < \bar{\alpha}_1 < \bar{\alpha}_2 < \alpha_2$, and if, moreover, the first integral to be evaluated in (2), namely:

$$\int_a^b f_\alpha(x, \alpha)\,dx,$$

converges uniformly in the interval $\bar{\alpha}_1 \leqq \alpha \leqq \bar{\alpha}_2$, the above proof will hold for all values of α' such that $\alpha_1 < \alpha' < \alpha_2$.

Cambridge, Massachusetts,
November 3, 1907.